Far East Journal of Theoretical Statistics

http://www.pphmj.com
http://dx.doi.org/10.17654/0972086323001
Volume 67, 2023, Pages 1-14 **P-ISSN: 0972-0863**

ON THE REGRESSION ESTIMATION FROM $\tilde{\rho}$-MIXING SAMPLES

Mounir Arfi

Department of Statistics
College of Science
Sultan Qaboos University
P.O. Box 50 Al-Khod
Postal Code No: 123
Muscat, Sultanate of Oman
e-mail: arfi@squ.edu.om
mounir.arfi@gmail.com

Abstract

We give the rate of the uniform convergence for the kernel estimate of the regression function over a sequence of compact sets which increases to $\mathbb{R}^d$ when n grows to infinity and the observed process is $\tilde{\rho}$-mixing. The used estimator for the regression function is the kernel estimator proposed by Nadaraya [10] and Watson [12].

Received: October 19, 2022; Revised: November 13, 2022; Accepted: December 15, 2022
2020 Mathematics Subject Classification: 62G05, 62G08.
Keywords and phrases: regression, kernel estimate, $\tilde{\rho}$-mixing.

How to cite this article: Mounir Arfi, On the regression estimation from $\tilde{\rho}$-mixing samples, Far East Journal of Theoretical Statistics 67 (2023), 1-14.
http://dx.doi.org/10.17654/0972086323001

Published Online: December 23, 2022

1. Introduction

Let $(X_t, Y_t)_t \in \mathbb{N}$ be a strictly stationary process, where (X_t, Y_t) take values in $\mathbb{R}^d \times \mathbb{R}$ and distributed as (X, Y). Suppose that a segment of data $(X_t, Y_t)_{t=1}^n$ has been observed.

We are interested in the study of the convergence rate for a kernel estimate of the regression function, known as

$$r(x) = E(Y_t \mid X_t = x), \quad t \in \mathbb{N}.$$

A natural estimator for the function $r(\cdot)$ is given by

$$r_n(x) = \frac{\sum_{t=1}^{n} Y_t K\left(\frac{x - X_t}{h_n}\right)}{\sum_{t=1}^{n} K\left(\frac{x - X_t}{h_n}\right)}, \quad \forall x \in E,$$

where E stands for the subset $\{x \in \mathbb{R},\ f(x) > 0\}$, f being the density of the process (X_t) and (h_n) is a positive sequence of real numbers such that $h_n \to 0$ and $nh_n^d \to \infty$, when $n \to \infty$.

K is a Parzen-Rosenblatt kernel type in the sense of a bounded function satisfying

$$\int_{\mathbb{R}} K(x)dx = 1 \quad \text{and} \quad \lim_{\|x\| \to \infty} \|x\| K(x) = 0.$$

Moreover, it is assumed to be strictly positive and with bounded variation.

2. Preliminaries and Assumptions

Let $(\Omega, \mathcal{F}, \mathcal{P})$ be a probability space.

Let $(X_t, t \in \mathbb{N})$ be a sequence of random variables. Then we define

$$\mathcal{F}_2 = \sigma(X_t, t \in S), \text{ where } S \text{ is a subset of } \mathbb{N}.$$

Therefore, given the σ-algebra $\mathcal{B}$ and $\mathcal{R}$ in $\mathcal{F}$, let

$$\rho(\mathcal{B}, \mathcal{R}) = \sup\{\mathrm{corr}(X, Y), X \in L_2(\mathcal{B}), Y \in L_2(\mathcal{R})\},$$

where

$$\mathrm{corr}(X, Y) = \frac{E(XY) - E(X)E(Y)}{\sqrt{\mathrm{var}(X)\,\mathrm{var}(Y)}}.$$

Bradley [5] introduced the following coefficients of dependence:

$$\tilde{\rho}(k) = \sup\{\rho(\mathcal{F}_s, \mathcal{F}_t)\}, \quad k \geq 0,$$

where the supermum is taken over all finite subsets $S, T \subset \mathbb{N}$ such that $\mathrm{dist}(S, T) \geq k$.

Obviously,

$$0 \leq \tilde{\rho}(k+1) \leq \tilde{\rho}(k) \leq 1, \quad k \geq 0 \quad \text{and} \quad \tilde{\rho}(0) = 1.$$

Definition 2.1. A random sequence of variables $(X_t, t \geq 1)$ is said to be a $\tilde{\rho}$-*mixing sequence* if there exists a $k \in \mathbb{N}$ such that $\tilde{\rho}(k) < 1$.

Without loss of generality, we may assume that $(X_t, t \geq 1)$ is such that $\tilde{\rho}(1) < 1$ (see Bryc and Smolenski [7]).

In the study of $\tilde{\rho}$-mixing sequences, we refer to Bradley [5, 6] for the central limit theorem, Bryc and Smolenski [7] for moment inequalities and almost sure convergence, Peligrad and Gut [9], Shixin [11] for almost sure results, Arfi [2] for almost sure convergence of the mode function, and Arfi [3] for an estimation of the hazard function.

We make use of the following assumptions:

A1. The observed process $(X_t, t \in \mathbb{N})$ is stationary and $\tilde{\rho}$-mixing.

A2.

$$\exists \Gamma < \infty, \quad \forall x \in \mathbb{R}^d, \quad f(x) \leq \Gamma$$

and

$$\exists \gamma_n > 0; \quad \forall x \in C_n, \quad f(x) \geq \gamma_n,$$

where C_n is a sequence of compact sets such that $C_n = \{x : \| x \| \leq c_n\}$ with $c_n \to \infty$.

A3. $\exists b \geq 2, \exists M < \infty$ such that $E(| Y |^b) < M$.

A4. $\exists V < \infty, \forall x \in \mathbb{R}^d$, $E[(Y - r(x))^2 \,|\, X = x] \leq V$.

A5. The density f is twice differentiable and its second derivatives are bounded on $\mathbb{R}^d$.

A6. The kernel K is Lipschitz of ratio L_k, that is $| K(x) - K(y) | \leq L_k \| x - y \|^k$.

3. Main Result

Theorem 3.1. *Assuming that the assumptions A1 through A6 hold, the function r is Lipschitz, bounded on* $\mathbb{R}^d$ *and that the bandwidth sequence* (h_n) *satisfies with* y_n :

$$nh_n^d y_n^{-1} n^{-\delta} = \infty(Logn), \quad n^\delta \gamma_n^{-1} \to 0 \text{ and } n^\delta \gamma_n^{-1} h_n^{-d} y_n^{-b/2} \to 0, \quad n \to \infty,$$

where y_n *is an unbounded and nondecreasing sequence chosen so that*

$$1 \leq y_n \leq n/2.$$

If the kernel K is even with $\int z^2 K(z)dz < \infty$ *for* $z = (z_1, ..., z_d)$ *and if there exists a constant D such that* $\gamma_n^{-1} y_n n^\delta h_n^d < D$, *then*

$$n^\delta \sup_{\| x \| \leq c_n} | r_n(x) - r(x) | = O(1) \text{ a.s. } n \to \infty.$$

4. Preliminary Results

For practical reasons, we make the following decomposition:

$$r_n(x) - r(x) = \frac{1}{f(x)}\{[g_n(x) - r(x)f(x)] - r_n(x)[f_n(x) - f(x)]\},$$

where

$$g_n(x) = \frac{1}{nh_n^d}\sum_{t=1}^{n} Y_t K\left(\frac{x - X_t}{h_n}\right)$$

and

$$f_n(x) = \frac{1}{nh_n^d}\sum_{t=1}^{n} K\left(\frac{x - X_t}{h_n}\right).$$

This leads to

$$\sup_{x\in C_n} |r_n(x) - r(x)|$$

$$= \frac{1}{f(x)}\{\sup_{x\in C_n} |g_n(x) - r(x)f(x)| + \sup_{x\in C_n} |r_n(x)||f_n(x) - f(x)|\}.$$

Then if

$$\sup_{x\in C_n} |r_n(x)| \le y_n \ a.s.,$$

we obtain

$$\sup_{x\in C_n} |r_n(x) - r(x)|$$

$$= \gamma_n^{-1}\{\sup_{x\in C_n} |g_n(x) - r(x)f(x)| + y_n \sup_{x\in C_n} |f_n(x) - f(x)|\}.$$

Lemma 4.1. *Under the hypotheses of Theorem* 3.1,

$$\gamma_n^{-1} n^{\delta} \sup_{x\in C_n} |g_n(x) - Eg_n(x)| \to 0 \ a.s. \ n \to \infty.$$

Proof. Because of the possible large values for Y_t, we use a truncation technique which consists in decomposing g_n in g_n^+ and g_n^-, where

$$g_n^+(x) = \frac{1}{nh_n^d}\sum_{t=1}^{n} Y_t \mathbb{I}_{[|Y_t|>y_n]} K\left(\frac{x - X_t}{h_n}\right)$$

and $g_n^-(x) = g_n(x) - g_n^+(x)$, with an unbounded sequence y_n defined as in Theorem 3.1.

We start by showing that

$$\gamma_n^{-1} n^{\delta} \sup_{\|x\|\leq c_n} |g_n^-(x) - Eg_n^-(x)| \to 0 \ a.s. \ n \to \infty.$$

To this end, we write

$$g_n^-(x) - Eg_n^-(x) = \sum_{t=1}^{n} \varphi_t$$

with

$$\varphi_t = \frac{1}{nh_n^d}\left\{Y_t \mathbb{I}_{[|Y_t|\leq y_n]} K\left(\frac{x - X_t}{h_n}\right) - E\left[Y_t \mathbb{I}_{[|Y_t|\leq y_n]} K\left(\frac{x - X_t}{h_n}\right)\right]\right\}.$$

Therefore, $E(\varphi_t) = 0$;

$|\varphi_t| \leq \frac{2K_1 y_n}{nh_n^d} = d_n$, where K_1 is an upper bound of K, which permits to write

$$E|\varphi_t| \leq \frac{2\Gamma}{n} E\left|\frac{Y_t}{h_n^d} \mathbb{I}_{[|Y_t|\leq y_n]} K\left(\frac{x - X_t}{h_n}\right)\right|$$

$$\leq \frac{2\Gamma}{n}\int \frac{E(|Y_t|/X_t = u)}{h_n^d} K\left(\frac{x - u}{h_n}\right) du.$$

By Schwartz inequality and the assumption A4, we have

$$E|\varphi_t| \le \frac{2\Gamma}{n}\int \frac{(r^2(u)+V)^{1/2}}{h_n^d} K\left(\frac{x-u}{h_n}\right)du \le \tau_1 n^{-1},$$

where τ_1 is a positive constant.

Now, similar arguments provide

$$E(\varphi_t^2) \le \frac{2\Gamma}{n^2}\int \frac{(r^2(u)+V)}{h_n^{2d}} K\left(\frac{x-u}{h_n}\right)du \le vn^{-2}h_n^{-d},$$

where v is a positive constant.

Next, we write

$$\sum_{n=1}^{\infty} P(\gamma_n^{-1}n^{\delta}|\, g_n^{-}(x) - Eg_n^{-}(x)\,| > \varepsilon) = \sum_{n=1}^{\infty} P\left(\gamma_n^{-1}n^{\delta}\left|\sum_{t=1}^{n}\varphi_t\right| > \varepsilon\right)$$

and for $\alpha > 1$ and $1 \le t \le n$,

$$W_{nt} = \varphi_t \mathbb{I}_{(|\varphi_t| \le n^{\alpha})}, \quad Z_{nt} = \varphi_t \mathbb{I}_{(|\varphi_t| > n^{\alpha})}.$$

Then

$$\left|\sum_{t=1}^{n}\varphi_t\right| \le \left|\sum_{t=1}^{n}(W_{nt} - EW_{nt})\right| + \left|\sum_{t=1}^{n}Z_{nt}\right| + \left|\sum_{t=1}^{n}EW_{nt}\right|. \tag{1}$$

We need only to show the following:

$$\sum_{n=1}^{\infty} P\left(\gamma_n^{-1}n^{\delta}\left|\sum_{t=1}^{n}(W_{nt} - EW_{nt})\right| > \varepsilon n^{\alpha}/3\right) < \infty, \tag{2}$$

$$\sum_{n=1}^{\infty} P\left(\gamma_n^{-1}n^{\delta}\left|\sum_{t=1}^{n}Z_{nt}\right| > \varepsilon n^{\alpha}/3\right) < \infty, \tag{3}$$

$$\frac{\gamma_n^{-1} n^{\delta} \left| \sum_{t=1}^{n} EW_{nt} \right|}{n^{\alpha}} \to 0, \quad n \to \infty. \tag{4}$$

We start by showing (2).

The Markov inequality and Chebyshev's inequality lead to

$$\sum_{n=1}^{\infty} P\left(\gamma_n^{-1} n^{\delta} \left| \sum_{t=1}^{n} (W_{nt} - EW_{nt}) \right| > \varepsilon n^{\alpha}/3 \right)$$

$$\leq c_1 \sum_{n=1}^{\infty} \sum_{t=1}^{n} \gamma_n^{-1} n^{\delta} E| W_{nt} |^2 / n^{2\alpha}$$

$$\leq c_2 \sum_{n=1}^{\infty} \gamma_n^{-1} n^{\delta} h_n^{-d} n^{-1-2\alpha} < \infty$$

if we choose

$$\gamma_n = n^{-a} \text{ for } a > 0 \text{ and } h_n = n^{-\tau} \text{ for } (a + \alpha + \delta)/d < \tau < 1/2.$$

The Borel-Cantelli lemma allows to conclude for (2).

Now, we show (3).

Note that

$$\left(\left| \sum_{t=1}^{n} Z_{nt} \right| > \varepsilon n^{\alpha}/3 \right) \subset \bigcup_{t=1}^{n} (| \varphi_t | > n^{\alpha})$$

and hence

$$\sum_{n=1}^{\infty} P\left(\gamma_n^{-1} n^{\delta} \left| \sum_{t=1}^{n} Z_{nt} \right| > \varepsilon n^{\alpha}/3 \right) \leq \sum_{n=1}^{\infty} n P(\gamma_n^{-1} n^{\delta} | \varphi_t | > n^{\alpha})$$

$$\le \sum_{n=1}^{\infty} n\gamma_n^{-1} n^{\delta} E|\varphi_t|^2 / n^{2\alpha}$$

$$\le c_3 \sum_{n=1}^{\infty} n^{-1-2\alpha}\gamma_n^{-1} n^{\delta} h_n^{-d} < \infty,$$

with the following:

$$\gamma_n = n^{-a} \text{ for } a > 0, \quad h_n = n^{-\tau} \text{ for } (a + \alpha + \delta)/d < \tau < 1/2,$$

and c_3 being a positive constant.

Lastly, we show that (4) holds.

We can write

$$\gamma_n^{-1} n^{\delta} n^{-\alpha} \left| \sum_{t=1}^{n} EW_{nt} \right| \le \gamma_n^{-1} n^{\delta} n^{-\alpha} \sum_{t=1}^{n} |EW_{nt}|$$

$$\le \gamma_n^{-1} n^{\delta} n^{-\alpha} \sum_{t=1}^{n} E|\varphi_t| \mathbb{I}_{(|\varphi_t| > n^{\alpha})}$$

$$= n^{a+\delta+1-\alpha} E|\varphi_t| \mathbb{I}_{(|\varphi_t| > n^{\alpha})} \to 0, \; n \to \infty$$

with the choice $\gamma_n = n^{-a}$ and $\alpha > 2 + a + \delta$.

Next, we cover C_n by μ_n^d spheres in the shape of

$$\{x : \| x - x_{nj} \| \le c_n \mu_n^{-1}\} \text{ with } 1 \le j \le \mu_n^d$$

and we can make the following decomposition:

$$| g_n^-(x) - Eg_n^-(x) |$$

$$\le | g_n^-(x) - g_n^-(x_{jn}) | + | g_n^-(x_{jn}) - Eg_n^-(x_{jn}) | + | Eg_n^-(x_{jn}) - Eg_n^-(x) |.$$

Therefore,

$$n^{\delta}|\ g_n^{-}(x) - g_n^{-}(x_{jn})\ | \le \frac{n^{\delta}}{nh_n^d}\ y_n \sum_{t=1}^{n}\left| K\left(\frac{x - X_t}{h_n}\right) - K\left(\frac{x_{jn} - X_t}{h_n}\right)\right|.$$

Since the kernel K is Lipschitz, we have

$$n^{\delta}|\ g_n^{-}(x) - g_n^{-}(x_{jn})\ | \le L_K\, y_n n^{\delta}\ \frac{1}{h_n^{d+k}} \|\ x - x_{jn}\ \|^k$$

$$\le L_K\, y_n n^{\delta} h_n^{-d-k} c_n^k \mu_n^{-k}\ = 1/Logn.$$

If we choose

$$\mu_n\ = L_K^{1/k}\, y_n^{1/k} h_n^{-\left(\frac{d}{k}+1\right)} c_n n^{\delta/k} (Logn)^{1/k}\ \to \infty,$$

we obtain

$$\sup_{x\in C_n}\ |\ g_n^{-}(x) - Eg_n^{-}(x)\ | \le \sup_{1\le j\le \mu_n^d}\ |\ g_n^{-}(x_{jn}) - Eg_n^{-}(x_{jn})\ | + \left(\frac{2}{Logn}\right)$$

so that for all $n \ge n_1(\varepsilon),\ \forall \varepsilon_n > 0,$ we have

$$P\left(\sup_{x\in C_n}\left|\sum_{t=1}^{n}\varphi_t\right| > 2\varepsilon_n\right) \le \sum_{j=1}^{\mu_n} P(|\ g_n^{-}(x_{jn}) - Eg_n^{-}(x_{jn})\ | > \varepsilon_n).$$

Now using similar decomposition as in (1) μ_n times; the use of $(\mu_n^d n^{\delta} n^{\alpha})$ instead of n^{α} permits to conclude that

$$n^{\delta}\gamma_n^{-1} \sup_{\|\ x\ \|\le c_n}\ |\ g_n^{-}(x) - Eg_n^{-}(x)\ | \to 0,\ a.s.\ n \to \infty.$$

It remains to show that

$$n^{\delta}\gamma_n^{-1} \sup_{\|\ x\ \|\le c_n}\ |\ g_n^{+}(x) - Eg_n^{+}(x)\ | \to 0,\ a.s.\ n \to \infty.$$

For the purpose, we write

$$n^{\delta}\gamma_n^{-1} \sup_{\| x \|\le c_n} | g_n^{+}(x) - Eg_n^{+}(x) | \le E_n + F_n,$$

where

$$E_n = \frac{n^{\delta}\gamma_n^{-1}}{nh_n^d} \sup_{\| x \|\le c_n} \left| \sum_{t=1}^{n} Y_t \mathbb{I}_{(| Y_t | > y_n)} K\left(\frac{x - X_t}{h_n}\right) \right|$$

and we have

$$(E_n \neq 0) \subset \{\exists t_0 \in [1, 2, 3, ..., n] \text{ such that } | Y_{t_0} | > y_n\}$$

and we can write

$$(E_n \neq 0) \subset \bigcup_{t=1}^{n} \{| Y_t | > y_n\},$$

$$P(E_n \neq 0) \le \sum_{t=1}^{n} P(| Y_t | > y_n) = nP(| Y_t | > y_n),$$

$$\sum_n P(E_n \neq 0) \le \sum_n P(| Y_t | > y_n) \le \sum_n ny_n^{-b} E| Y |^b,$$

$$\sum_n P(E_n \neq 0) \le c_4 \sum_n ny_n^{-b} < \infty,$$

where c_4 is a positive constant.

Then $E_n \to 0,\ a.s.,\ n \to \infty$ and $\sup_{1 \le t \le n} | Y_t | \le y_n\ a.s.$

The kernel K being strictly positive, we conclude that $| r_n(x) | \le y_n\ a.s.$

Moreover,

$$F_n = \frac{n^{\delta}}{\gamma_n nh_n^d} \sup_{\| x \|\le c_n} \left| \sum_{t=1}^{n} E\left[Y_t \mathbb{I}_{[| Y_t | > y_n]} K\left(\frac{x - X_t}{h_n}\right)\right] \right|,$$

$$F_n \le \frac{n^{\delta}}{\gamma_n h_n^d} K_1 E[|Y| \mathbb{I}_{|Y|>y_n}],$$

$$F_n \le \frac{n^{\delta}}{\gamma_n h_n^d} K_1 (E(Y^2))^{1/2} (P[|Y| > y_n])^{1/2}$$

$$\le c_5 n^{\delta} \gamma_n^{-1} h_n^{-d} y_n^{-b/2} \to 0, \quad n \to \infty,$$

where c_5 is a positive constant.

Lemma 4.2. *Under assumptions of Theorem* 3.1,

$$n^{\delta} \gamma_n^{-1} \sup_{x \in \mathbb{R}^d} |Eg_n(x) - r(x) f(x)| \to 0, \quad n \to \infty.$$

Proof.

$$Eg_n(x) - r(x) f(x) = \frac{1}{nh_n^d} E\left\{\sum_{t=1}^{n} Y_t K\left(\frac{x - X_t}{h_n}\right)\right\} - r(x) f(x),$$

$$Eg_n(x) - r(x) f(x) = \frac{1}{h_n^d} \int_{\mathbb{R}^d} r(u) K\left(\frac{x - u}{h_n}\right) f(u) du - r(x) f(x).$$

Writing $z = (x - u)/h_n$, we obtain

$$Eg_n(x) - r(x) f(x) = \int_{\mathbb{R}^d} [r(x - zh_n) - r(x)] K(z) f(x - zh_n) dz$$

$$+ r(x) \int K(z) [f(x - zh_n) - f(x)] dz.$$

Assuming that the function $r(\cdot)$ is Lipschitz of ratio 1 and order 1, we have

$$\left| \int_{\mathbb{R}^d} [r(x - zh_n) - r(x)] K(z) f(x - zh_n) dz \right| \le h_n \Gamma \int |Z| K(z) dz.$$

Now a Taylor expansion, the Bochner lemma and the fact that the function r is bounded permit to conclude that

$$n^{\delta}\gamma_n^{-1} \sup_{x \in \mathbb{R}^d} |Eg_n(x) - r(x)f(x)| \to 0, \quad n \to \infty.$$

Lemma 4.3. *Under assumptions of Theorem* 3.1,

$$\lim_{n\to\infty} \frac{y_n n^{\delta}}{\gamma_n} \sup_{\| x \| \le c_n} | f_n(x) - Ef_n(x)| = 0 \ a.s.$$

Proof. This is a particular case of Lemma 4.1 when $Y_t = 1$ and $\varepsilon = \varepsilon_0 \gamma_n n^{\delta} y_n^{-1}$ for a certain $\varepsilon_0 > 0$.

Lemma 4.4. *Under assumptions of Theorem* 3.1,

$$\lim_{n\to\infty} \frac{y_n n^{\delta}}{\gamma_n} \sup_{x \in \mathbb{R}^d} | Ef_n(x) - f(x)| = 0.$$

Proof. We write

$$Ef_n(x) - f(x) = \frac{1}{h_n^d} \int [f(u) - f(x)] K\left(\frac{u - x}{h_n}\right) du.$$

A Taylor expansion, the hypotheses of Theorem 3.1 and the Bochner lemma permit to conclude.

Acknowledgement

The author is thankful to the referees for their comments and criticisms.

References

[1] M. Arfi, Sur la regression non paramétrique d'un processus stationnaire mélangeant ou ergodique, Thèse de Doctorat de l' Université Paris 6, 1996.

[2] M. Arfi, Estimation of the mode function for $\tilde{\rho}$-mixing observations, J. Stat. Appl. Pro. 1(3) (2012), 183-192.

[3] M. Arfi, Nonparametric estimation for the hazard function, Comm. Statist. Theory Methods 42 (2013), 2543-2550.

[4] D. Bosq, Nonparametric statistics for stochastic processes, Lecture Notes in Statistics, 110, Springer-Verlag, 1996.

[5] R. C. Bradley, Equivalent mixing conditions of random fields, Technical Report 336, Center for Stochastic Processes, University of North Carolina, Chapel Hill, 1990.

[6] R. C. Bradley, On the spectral density and asymptotic normality of weakly dependent random fields, J. Theoret. Probab. 5 (1992), 355-373.

[7] W. Bryc and W. Smolenski, Moment conditions for almost sure convergence of weakly correlated random variables, Proc. Amer. Math. Soc. 119 (1993), 629-635.

[8] G. Collomb, Nonparametric regression: an up-to-date bibliography, Statistics 16 (1985), 309-324.

[9] M. Peligrad and A. Gut, Almost sure results for a class of dependent random variables, J. Theoret. Probab. 12 (1999), 87-104.

[10] E. A. Nadaraya, On estimating regression, Theory Probab. Appl. 9 (1964), 141-142.

[11] G. Shixin, Almost sure convergence for $\tilde{\rho}$-mixing random variables sequences, Statist. Probab. Lett. 67 (2004), 289-298.

[12] G. S. Watson, Smooth regression analysis, Sankhya 26 (1964), 359-372.